GLIMPSES OF GENIUS

Mathematicians and historians piece together a puzzle that Archimedes pondered

BY ERICA KLARREICH

At the start of the 20th century, a Danish mathematical historian named Johan Ludvig Heiberg made a once-in-a-lifetime find. Tucked away in the library of a monastery in Istanbul was a medieval parchment containing copies of the works of the ancient Greek mathematician Archimedes, including two never-before-seen essays. To mathematicians' astonishment, one of the new essays contained many of the key ideas of calculus, a subject supposedly invented two millennia after Archimedes' time. The essay caused a sensation and landed Heiberg's discovery on the front page of a 1907 New York Times.

The other new essay, by contrast, mystified mathematicians. A fragment of a treatise called the Stomachion, it appeared to be nothing more than a description of a puzzle that might have been a children's toy. Mathematicians wondered why Archimedes, whose other works were so monumental, should have spent his time on something so frivolous.

The Stomachion fragment offered only tantalizing glimpses into Archimedes thinking. The parchment, probably first written on in the 10th century in Constantinople, is a palimpsest—a document whose surface has been scraped and reused. In 1204, the Fourth Crusade sacked Constantinople, and shortly thereafter monks unbound the Archimedes's parchment, did their best to erase the mathematical text, and recycled the volume as a Christian prayer book. Only the beginning pages of the Stomachion made it into the new book, and on those pages, the underlying math text was faint and hard to read.

RECYCLING PROGRAM — In the 13th century, monks reused a copy of Archimedes' work by writing prayer texts (horizontal lines) over the underlying mathematical text (vertical lines).

Armed with only a magnifying glass, Heiberg managed to make out a large portion of the palimpsest. What he read, however, offered few clues to Archimedes' interest in the puzzle. And before other scholars could examine the faded script, and perhaps catch something Heiberg had missed, the parchment was stolen. It vanished into obscurity for more than 8 decades.

In 1998, the Archimedes palimpsest suddenly resurfaced from a private collection. An anonymous U.S. billionaire bought it for $2 million at Christie's auction house in New York and made it available to scholars. Mathematical historians have now come up with a guess about what Archimedes was thinking 2,200 years ago.

Mathematicians are also studying the puzzle for its own sake. Its underlying structure, they are finding, is anything but trivial.

A MULTITUDE OF SOLUTIONS The Stomachion puzzle consists of 14 pieces—all triangles, quadrilaterals, or 5-sided figures—that fit together like a tangram to make a square and many other shapes, including those of angular elephants, dancers, and sailboats. Archimedes was not the puzzle's inventor—it was around before his time, says Reviel Netz, a Stanford University math historian on the team of experts studying the palimpsest. It's possible, he says, that Archimedes was attracted to the puzzle after seeing it among his children's toys.

Heiberg and his contemporaries doubted that Archimedes had idled away his time making elephants, dancers, and sailboats. "I think Heiberg never succeeded in fathoming the purpose of this essay," says Nigel Wilson, a classics professor at Oxford University, who has been studying the palimpsest with Netz.

Wilson, Netz, and their collaborators have a greater challenge than Heiberg did in transcribing the palimpsest because the manuscript's condition has worsened in the intervening years. Not only has the ink faded, but mold has discolored or even eaten through the pages in many places.

During the decades that the parchment was missing, someone painted fake medieval illuminations on some pages in the palimpsest, probably to make the parchment look more valuable to a buyer. Luckily the text of the Stomachion wasn't touched.

Despite the palimpsest's deteriorating condition, high-tech digital-imaging techniques are now revealing some of the text that eluded Heiberg. In a recent examination of the Stomachion, Netz and Wilson made out a word that Heiberg had transcribed incorrectly. In the new reading, it's *plethos*, which comes from the same stem as *plethora* and translates roughly to "multitude" or "quantity."

The word suggested a novel possibility to Netz: Archimedes had been concerned not with the various shapes that can be made from the puzzle pieces but with how many different ways the pieces can be positioned to make a square. If Netz is correct, then Archimedes was interested in combinatorics, the branch of mathematics that deals with counting combinations and arrangements of objects. This field has come into its own only in the past half-century.

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TOUCHING ARCHIMEDES Netz wasn't sure whether it would have been feasible for Archimedes to count the number of square tilings that could be made from the puzzle pieces. Was it an easy calculation, he wondered, or something far too difficult to be computed by hand?

Netz asked Stanford mathematicians Persi Diaconis and Susan Holmes just how hard a problem it was. The pair started playing with a physical model of the Stomachion puzzle and discovered that counting the different tilings was surprisingly difficult.

"We thought, at first, we'd just be able to sit down and figure out in a day how many tilings there were," Holmes says. "Then, we realized that there are a lot, way more than you'd expect."

Together with mathematicians Ron Graham and Fan Chung of the University of California, San Diego, the pair studied the puzzle on and off for several months, spurred on by the knowledge that Archimedes had considered it important.

HOW MANY WAYS? — Classicists speculate that Archimedes counted the number of different ways these puzzle pieces can be rearranged to make a square.

"For me, it was an excuse to reach back and touch Archimedes in some way," Diaconis says. "It was exciting to think I might be wrestling with some of the same things he did."

The team determined that the pieces can be put together to make a square in no fewer than 268 different ways. That number of tilings, while large, could indeed be computed by hand. "Archimedes could have counted them," Diaconis says.

An independent computer calculation by computer scientist William Cutler of Elgin, Ill., also came up with 268 different tilings.

The mathematicians had answered Netz' question, but by then, Chung and Graham were hooked on the problem. While playing with the pieces, they had noticed interesting patterns that convinced them that there were more discoveries to be made about the Stomachion.

TILING TOURS The mathematicians noticed that the tilings always seem to contain a smaller, symmetrical shape, such as a square or isosceles triangle. If that smaller shape gets rotated or flipped, the result is a related tiling that differs from the first one only within the small shape.

Graham and Chung decided to try to visualize these relationships using a network, akin to the Internet, consisting of points with lines connecting some of them. In this network, each point represents a different tiling. A line connects two points if the two tilings are related to each other by rotation or flipping of an internal, symmetrical shape.

The advantage of translating the puzzle into a network is that Chung and Graham could then turn to an arsenal of theoretical tools. They started by asking the standard mathematical questions about networks: Are the points connected? What kinds of routes run through the lines of the network?

To their surprise, the pair found that the network splits into two chunks—a huge one containing 266 of the tilings and a tiny one containing the other 2.

"We didn't expect so much [of the network] to be connected, but once it was, it was a surprise that there were these two tilings outside the main galaxy," Graham says.

The pair of oddball tilings has considerably less symmetry than the others do. Each contains just one small trapezoid that can be flipped to get the other tiling.

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In the large chunk, Chung and Graham found a "traveling salesman" circuit—a path through the network that visits each point exactly once. In other words, starting with any one tiling, it's possible to go on a tour of all the tilings, except the two oddballs, by doing one simple flip or rotation after another.

To get from one tiling to another, however, it's not necessary to go through all the other tilings. Graham and Chung showed that the network has 11 degrees of separation—that is, to get from one tiling to another, at most 11 simple moves are necessary. For a randomly chosen pair of tilings, Graham estimates that an average of five moves suffices.

Returning from the network to the puzzle itself, the team became intrigued by a special property of the puzzle first observed by Athina Markopoulou, an electrical-engineering postdoctoral student at Stanford. If the Stomachion puzzle is placed on a 12-by-12 grid, then all the corners of the tiles lie on lattice points on the grid. So far, that's not too exciting—lots of puzzles have that property.

What is surprising is that no matter how the Stomachion tiles are rearranged to make a square, the corners still fall on lattice points. This is not generally true of square puzzles.

"There's something special about the tiles in the Stomachion puzzle," Graham says. As yet, the researchers have no clue to why the Stomachion has the lattice-point property.

A NOBLE LINEAGE Was Archimedes aware of all this structure? "I wouldn't put it past Archimedes to recognize this was a special puzzle," Wilson says.

According to Netz, however, it's unlikely that Archimedes constructed a network like Graham and Chung's. The entire idea of representing relationships using a network is very recent, he says, and wouldn't have occurred to a Greek mathematician.

There are some hints that Archimedes may have been pursuing geometric theorems by studying the Stomachion. In his essay, he mentions nearly complete puzzles. He may have been referring to puzzles in which all the pieces but one have been placed, leaving a hole, Holmes says. The hole would have the same area as the remaining piece, so Archimedes may have been using the puzzle to calculate areas of complicated shapes.

Diaconis speculates that Archimedes might have been using the Stomachion to construct a pictorial proof of some geometric result such as the Pythagorean theorem, the famous equation relating the lengths of the two shorter sides of a right triangle and the length of the hypotenuse. "I keep thinking some amateur mathematician should fiddle with those pieces and see if they can find the Pythagorean theorem somewhere," he says.

Netz and Wilson, on the other hand, are writing a paper arguing that Archimedes was concerned with the number of different tilings of the square. However, the precise arguments Archimedes made will probably remain in the realm of speculation unless more of the Stomachion essay comes to light. "It's well known that Archimedes is a bit unpredictable," Wilson says. "He's very clever, and ordinary mortals can't easily guess how he'll get from step A to step B."

The idea that Archimedes might have counted the number of tilings tickles Diaconis, who is a combinatorialist himself. "Combinatorics is the Johnny-come-lately of mathematics, and mathematicians don't treat it with the respect they give to geometry or number theory," he says. "This shows we have lineage too." ■